Competitively tight graphs

Suh-Ryung KIM[a,1], Jung Yeun LEE[b], Boram PARK[c,2], Yoshio SANO[d,3,*]

[a]*Department of Mathematics Education, Seoul National University, Seoul 151-742, Korea*
[b]*National Institute for Mathematical Sciences, Daejeon 305-390, Korea*
[c]*DIMACS, Rutgers University, Piscataway, NJ 08854, United States*
[d]*National Institute of Informatics, Tokyo 101-8430, Japan*

Abstract

The competition graph of a digraph D is a (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is the smallest number of such isolated vertices. Computing the competition number of a graph is an NP-hard problem in general and has been one of the important research problems in the study of competition graphs. Opsut [1982] showed that the competition number of a graph G is related to the edge clique cover number $\theta_E(G)$ of the graph G via $\theta_E(G) - |V(G)| + 2 \le k(G) \le \theta_E(G)$. We first show that for any positive integer m satisfying $2 \le m \le |V(G)|$, there exists a graph G with $k(G) = \theta_E(G) - |V(G)| + m$ and characterize a graph G satisfying $k(G) = \theta_E(G)$. We then focus on what we call *competitively tight graphs* G which satisfy the lower bound, i.e., $k(G) = \theta_E(G) - |V(G)| + 2$. We completely characterize the competitively tight graphs having at most two triangles. In addition, we provide a new upper bound for the competition number of a graph from which we derive a sufficient condition and a necessary condition for a graph to be competitively tight.

Keywords: Competition graph; Competition number; Edge clique cover; Upper bound; Competitively tight
2010 MSC: 05C75, 05C20

1. Introduction

Let D be a digraph. The *competition graph* of D, denoted by $C(D)$, is the (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. The notion of competition graph is due

[*]Corresponding author. *E-mail address:* `sano@nii.ac.jp`; `y.sano.math@gmail.com`

[1]This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (700-20110036).

[2]This work was supported by National Research Foundation of Korea Grant funded by the Korean Government, the Ministry of Education, Science and Technology (NRF-2011-357-C00004).

[3]The author was supported by JSPS Research Fellowships for Young Scientists.

to Cohen [1]. For any graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. From this observation, Roberts [9] defined the *competition number* $k(G)$ of a graph G to be the minimum integer k such that G together with k isolated vertices is the competition graph of an acyclic digraph.

It does not seem to be easy in general to compute the competition number $k(G)$ for a given graph G, as Opsut [8] showed that the computation of the competition number of a graph is an NP-hard problem. To compute exact values or give bounds for the competition numbers of graphs has been one of the foremost problems in the study of competition graphs (see [2] for a survey).

There is a well-known upper and lower bound for the competition numbers of arbitrary graphs due to Opsut [8]. A subset $S \subseteq V(G)$ of the vertex set of a graph G is called a *clique* of G if the subgraph $G[S]$ of G induced by S is a complete graph. For a clique S of a graph G and an edge e of G, we say *e is covered by S* if both of the endvertices of e are contained in S. An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is covered by some clique in the family. The *edge clique cover number* of a graph G, denoted $\theta_E(G)$, is the minimum size of an edge clique cover of G. Opsut showed the following.

Theorem 1.1 ([8, Propositions 5 and 7])**.** *For any graph G, $\theta_E(G) - |V(G)| + 2 \leq k(G) \leq \theta_E(G)$.*

We note that the upper bound in Theorem 1.1 can be rewritten as $\theta_E(G) - |V(G)| + |V(G)|$, which leads us to ask: For any integers m, n satisfying $2 \leq m \leq n$, does there exist a graph G such that $|V(G)| = n$ and $k(G) = \theta_E(G) - n + m$? The answer is yes by the following proposition:

Proposition 1.2. *For any integers m and n satisfying $2 \leq m \leq n$, there exists a graph G such that $|V(G)| = n$ and $k(G) = \theta_E(G) - n + m$.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ and G be a graph on V with the edge set consisting of the edges of the path $v_1 v_2 \cdots v_{n-m+1}$ and the edges of the complete graph with vertex set $\{v_{n-m+1}, \ldots, v_n\}$. Since G is chordal, $k(G) = 1$ (see [9]). It is easy to see that $\theta_E(G) = (n - m) + 1$. Thus $k(G) = \theta_E(G) - n + m$. □

The upper bound in Theorem 1.1 is obtained by only very special graphs as the following proposition states.

Proposition 1.3. *Let G be a graph with n vertices. Then the equality $k(G) = \theta_E(G)$ holds if and only if G is the complete graph K_n or G is the edgeless graph I_n.*

Proof. If $G = K_n$, then $k(G) = 1 = \theta_E(G)$. If $G = I_n$, then $k(G) = 0 = \theta_E(G)$. Suppose that $G \neq K_n$ and $G \neq I_n$. There exists an edge clique cover $\{S_1, \ldots, S_r\}$ of G, where $r = \theta_E(G)$. Note that $r \geq 1$ since $G \neq I_n$, and there exists a vertex $v \in V(G) \setminus S_1$ since $G \neq K_n$. Take vertices $z_2, \ldots, z_r$ not in G. We define a digraph D by $V(D) := V(G) \cup I^*$ and $A(D) := \{(x, v) : x \in S_1\} \cup A^*$, where I^* and A^* are defined as follows. If $r = 1$, then $I^* = \emptyset$. Otherwise, $I^* = \{z_2, \ldots, z_r\}$. If $r = 1$, then $A^* = \emptyset$. Otherwise, $A^* = \cup_{i=2}^{r} \{(x, z_i) : x \in S_i\}$. Thus we have $k(G) \leq r - 1$, which implies that $k(G) \neq \theta_E(G)$. Hence the claim is true. □

Then it is natural to ask: By which graphs is the lower bound in Theorem 1.1 achieved? To answer this question, we introduce the following notion.

Definition. A graph G is said to be *competitively tight* if it satisfies $k(G) = \theta_E(G) - |V(G)| + 2$.

We can show by the following result that any triangle-free graph G satisfying $|E(G)| \geq |V(G)| - 1$ is competitively tight.

Theorem 1.4 ([3, Theorem 8]). *If a graph G is triangle-free, then*

$$k(G) = \begin{cases} 0 & \text{if } |V(G)| = 1; \\ \max\{1, |E(G)| - |V(G)| + 2\} & \text{if } G \text{ has no isolated vertices}; \\ \max\{0, |E(G)| - |V(G)| + 2\} & \text{otherwise.} \end{cases}$$

By this theorem, we know that a triangle-free graph G without isolated vertices satisfying $|E(G)| \geq |V(G)| - 1 > 0$ has the competition number $|E(G)| - |V(G)| + 2$. We also know that a triangle-free graph G with isolated vertices satisfying $|E(G)| \geq |V(G)| - 2 \geq 0$ has the competition number $|E(G)| - |V(G)| + 2$. Since a triangle-free graph G satisfies $\theta_E(G) = |E(G)|$, the lower bound in Theorem 1.1 is achieved by any triangle-free graph G without isolated vertices satisfying $|E(G)| \geq |V(G)| - 1$ or any triangle-free graph G with isolated vertices satisfying $|E(G)| \geq |V(G)| - 2$. Conversely, if a graph G satisfies $|V(G)| = 1$, then $|E(G)| - |V(G)| + 2 = 1$ while $k(G) = 0$. If a triangle-free graph G has no isolated vertices and $|E(G)| \leq |V(G)| - 2$, then it cannot be competitively tight since $k(G) = 1$. If a triangle-free graph G has isolated vertices and $|E(G)| \leq |V(G)| - 3$, then it cannot be competitively tight since $k(G)$ is nonnegative. Thus the competitively tight triangle-free graphs can be characterized as follows:

Corollary 1.5. *A triangle-free graph G is competitively tight if and only if $|V(G)| \geq 2$ and G satisfies one of the following:*

(i) *G has no isolated vertices and $|E(G)| \geq |V(G)| - 1$;*

(ii) *G has isolated vertices and $|E(G)| \geq |V(G)| - 2$.*

2. Competitively tight graphs

We begin this section by presenting simple but useful results which show how to obtain competitively tight graphs from existing ones.

Lemma 2.1. *Let G be a graph and t be a nonnegative integer. Then we have $k(G \cup I_t) \geq k(G) - t$, and $k(G \cup I_t) = k(G) - t$ holds if and only if $0 \leq t \leq k(G)$.*

Proof. Suppose that $0 \leq t \leq k(G)$. Let $k = k(G)$. Then $G \cup I_k = (G \cup I_t) \cup I_{k-t}$ is the competition graph of an acyclic digraph. Thus we have $k(G \cup I_t) \leq k - t = k(G) - t$. To show that $k(G \cup I_t) \geq k(G) - t$, let $k' = k(G \cup I_t)$. Then $(G \cup I_t) \cup I_{k'} = G \cup I_{t+k'}$ is the competition graph of an acyclic digraph. Thus we have $k(G) \leq t + k'$, i.e. $k(G) - t \leq k' = k(G \cup I_t)$. Hence we have $k(G \cup I_t) = k(G) - t$. If $k(G) < t$, then we have $k(G) - t < 0 \leq k(G \cup I_t)$. Hence the lemma is true. $\square$

Proposition 2.2. *Suppose that a graph G is competitively tight. Let t be an integer such that $0 \leq t \leq k(G)$. Then the graph $G \cup I_t$ is competitively tight.*

Proof. Since G is competitively tight, $k(G) = \theta_E(G) - |V(G)| + 2$. Since $0 \leq t \leq k(G)$, by Lemma 2.1, $k(G \cup I_t) = k(G) - t$ holds. In addition, $\theta_E(G \cup I_t) = \theta_E(G)$ holds. Thus we have $k(G \cup I_t) = \theta_E(G) - |V(G)| + 2 - t = \theta_E(G \cup I_t) - |V(G \cup I_t)| + 2$. Hence $G \cup I_t$ is competitively tight. □

Proposition 2.3. *Suppose that a graph G ($\neq K_2$) is competitively tight and that G has a vertex v which is isolated or pendant. Then the graph $G - v$ obtained from G by deleting v is competitively tight.*

Proof. Since G is competitively tight, $k(G) = \theta_E(G) - |V(G)| + 2$. First, suppose that G has an isolated vertex v. Since v is isolated, $\theta_E(G - v) = \theta_E(G)$. By Lemma 2.1, $k(G) \geq k(G-v) - 1$. So we have $k(G-v) \leq k(G)+1 = \theta_E(G) - |V(G)| + 2 + 1 = \theta_E(G-v) - |V(G-v)| + 2$. By Theorem 1.1, $k(G-v) \geq \theta_E(G-v) - |V(G-v)| + 2$. Hence we have $k(G-v) = \theta_E(G-v) - |V(G-v)| + 2$, i.e., $G-v$ is competitively tight. Second, suppose that G has a pendant vertex v. Note that $\theta_E(G - v) = \theta_E(G) - 1$ and $|V(G - v)| = |V(G)| - 1$. Obviously $k(G - v) = k(G)$ when $G \neq K_2$. Hence we have $k(G - v) = \theta_E(G - v) - |V(G - v)| + 2$, i.e., $G - v$ is competitively tight. □

Due to Propositions 2.2 and 2.3, when we consider competitively tight graphs G, we may assume that the minimum degree of G is at least two.

We now begin the examination of competitively tight graphs which are not triangle-free. To this end, we recall several results of Kim and Roberts [6] which determine the competition numbers of various graphs with triangles of varying complexity. They found the competition number of a graph with exactly one triangle as the following theorem illustrates.

Theorem 2.4 ([6, Corollary 7])**.** *Suppose that a graph G is connected and has exactly one triangle. Then*

$$k(G) = \begin{cases} |E(G)| - |V(G)| & \textit{if } G \textit{ has a cycle of length at least four;} \\ |E(G)| - |V(G)| + 1 & \textit{otherwise.} \end{cases}$$

Let G be a connected graph with exactly one triangle. Then $\theta_E(G) = |E(G)| - 2$. If G has a cycle of length at least four, then $k(G) = \theta_E(G) - |V(G)| + 2$ by Theorem 2.4. Otherwise, $k(G) = \theta_E(G) - |V(G)| + 3$ by the same theorem. Now we have a characterization for the connected competitively tight graphs with exactly one triangle.

Proposition 2.5. *A connected graph with exactly one triangle is competitively tight if and only if it has a cycle of length at least four.*

Kim and Roberts [6] also determined the competition number of a graph with exactly two triangles. To do so, they defined $\mathrm{VC}(G)$ for a graph G as

$$\mathrm{VC}(G) := \{v \in V(G) : v \text{ is a vertex on a cycle of } G\}.$$

Let $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) be the family of graphs that can be obtained from Graph I (resp. one of the Graphs II-V) in Figure 1 by subdividing edges except those on triangles.

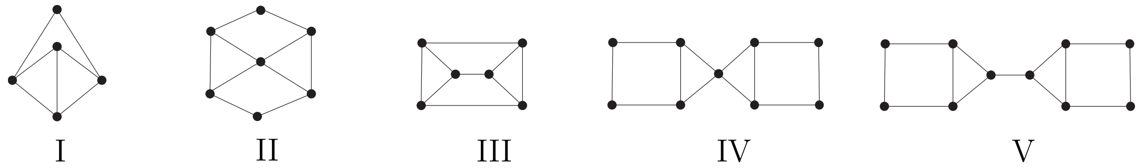

Figure 1: Graphs with exactly two triangles

Theorem 2.6 ([6, Theorem 9])**.** *Suppose that a connected graph G has exactly two triangles which share one of their edges. Then*

(a) $k(G) = |E(G)| - |V(G)|$ *if G is chordal or if the subgraph induced by* $\mathrm{VC}(G)$ *is in $\mathcal{G}_1$, and*

(b) $k(G) = |E(G)| - |V(G)| - 1$ *otherwise.*

Theorem 2.7 ([6, Theorem 10])**.** *Suppose that a connected graph G has exactly two triangles which are edge-disjoint. Then*

(a) $k(G) = |E(G)| - |V(G)|$ *if G is chordal,*

(b) $k(G) = |E(G)| - |V(G)| - 1$ *if G has exactly one cycle of length at least four as an induced subgraph or if the subgraph induced by* $\mathrm{VC}(G)$ *is in $\mathcal{G}_1 \cup \mathcal{G}_2$, and*

(c) $k(G) = |E(G)| - |V(G)| - 2$ *otherwise.*

From these two theorems, we may characterize the competitively tight graphs with exactly two triangles. A cycle of length at least four in a graph G is called a *hole* of G if it is an induced subgraph of G. The number of holes of a graph is closely related to its competition number (see [4, 7]).

Theorem 2.8. *A connected graph G with exactly two triangles is competitively tight if and only if G is not chordal and satisfies one of the following:*

(i) *the two triangles share one of their edges and the subgraph induced by* $\mathrm{VC}(G)$ *is not in $\mathcal{G}_1$;*

(ii) *the two triangles are edge-disjoint, G contains at least two holes, and the subgraph induced by* $\mathrm{VC}(G)$ *is not in $\mathcal{G}_1 \cup \mathcal{G}_2$.*

Proof. Let Δ_1 and Δ_2 be the two triangles of G. If Δ_1 and Δ_2 share an edge, then $\theta_E(G) = |E(G)| - 3$. If Δ_1 and Δ_2 are edge-disjoint, then $\theta_E(G) = |E(G)| - 4$. First, we show the "if" part. If (i) holds, then G satisfies the hypothesis of Theorem 2.6 (b) and so $k(G) = |E(G)| - |V(G)| - 1 = \theta_E(G) - |V(G)| + 2$. If (ii) holds, then G satisfies the hypothesis of Theorem 2.7 (c), $k(G) = |E(G)| - |V(G)| - 2 = \theta_E(G) - |V(G)| + 2$. Second, we show the "only if" part by contradiction. Suppose that G is chordal. Then, by Theorems 2.6 and 2.7, $k(G) = |E(G)| - |V(G)|$ or $|E(G)| - |V(G)| - 1$, none of which equals $\theta_E(G) - |V(G)| + 2$. Thus if G is chordal, then G is not competitively tight. Suppose that neither (i) nor (ii) holds. We consider the case where Δ_1 and Δ_2 share an edge. Then, by Theorem 2.6 (a), $\mathrm{VC}(G)$ induces a graph in $\mathcal{G}_1$ and so

$k(G) = |E(G)| - |V(G)|$, which does not equal $(|E(G)|-3) - |V(G)|+2 = \theta_E(G) - |V(G)|+2$. Now we consider the case where Δ_1 and Δ_2 are edge-disjoint. Then G contains exactly one hole or VC(G) induces a graph in $\mathcal{G}_1 \cup \mathcal{G}_2$. By Theorem 2.7 (b), $k(G) = |E(G)| - |V(G)| - 1 \neq (|E(G)|-4) - |V(G)|+2 = \theta_E(G) - |V(G)|+2$. □

It does not seem to be easy to characterize the competitively tight graph with exactly three triangles. Yet, we can show that there exists a competitively tight graph with exactly n triangles for each nonnegative integer n. We first give a new upper bound which improves the one given in Theorem 1.1. Let G be a graph and F be a subset of the edge set of G. We denote by $\theta_E(F;G)$ the minimum size of a family $\mathcal{S}$ of cliques of G such that each edge in F is covered by some clique in the family $\mathcal{S}$ (cf. [10]). We also need to introduce some notations. For a graph G, we define

$$\begin{aligned} E_{\triangle}(G) &:= \{e \in E(G) : e \text{ is contained in a triangle in } G\}, \\ \overline{E}_{\triangle}(G) &:= \{e \in E(G) : e \text{ is not contained in any triangle in } G\}. \end{aligned}$$

Note that $E_{\triangle}(G) \cup \overline{E}_{\triangle}(G) = E(G)$ and $E_{\triangle}(G) \cap \overline{E}_{\triangle}(G) = \emptyset$, and we can easily check the following lemma from the definitions.

Lemma 2.9. *For any graph G, $\theta_E(G) = \theta_E(E_{\triangle}(G);G) + |\overline{E}_{\triangle}(G)|$.*

Now we present a new upper bound for the competition number of a graph.

Theorem 2.10. *For any graph G,*

$$k(G) \leq \theta_E(E_{\triangle}(G);G) + \max\{\min\{1, |\overline{E}_{\triangle}(G)|\}, |\overline{E}_{\triangle}(G)| - |V(G)| + 2\}.$$

Proof. Let H be the graph obtained from G by deleting the edges in $E_{\triangle}(G)$, i.e., $H := G - E_{\triangle}(G)$. Then H is triangle-free and so, by Theorem 1.4,

$$k(H) \leq \max\{\min\{1, |E(H)|\}, |E(H)| - |V(H)| + 2\}.$$

Since $V(H) = V(G)$, $E(H) = \overline{E}_{\triangle}(G)$, the above inequality is equivalent to

$$k(H) \leq \max\{\min\{1, |\overline{E}_{\triangle}(G)|\}, |\overline{E}_{\triangle}(G)| - |V(G)| + 2\}.$$

Let D^- be an acyclic digraph such that $C(D^-) = H \cup I_{k(H)}$. Let $\mathcal{S}$ be a family of cliques of G of size $\theta_E(E_{\triangle}(G);G)$ such that each edge in $E_{\triangle}(G)$ is covered by some clique in $\mathcal{S}$. We define a digraph D by

$$V(D) := V(D^-) \cup \{z_S : S \in \mathcal{S}\} \quad \text{and} \quad A(D) := A(D^-) \cup \bigcup_{S \in \mathcal{S}} \{(v, z_S) : v \in S\}.$$

Then D is acyclic, and $C(D) = G \cup I_{k(H)} \cup \{z_S : S \in \mathcal{S}\}$. Therefore

$$k(G) \leq |\mathcal{S}| + k(H) \leq \theta_E(E_{\triangle}(G);G) + \max\{\min\{1, |\overline{E}_{\triangle}(G)|\}, |\overline{E}_{\triangle}(G)| - |V(G)| + 2\}.$$

Thus the theorem is true. □

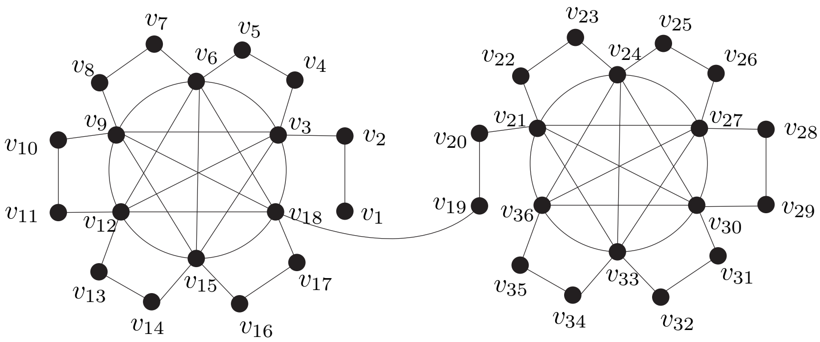

Figure 2: $G_{6,2}$.

Remark 2.11. The upper bound given in Theorem 2.10 is always better than the upper bound in Theorem 1.1. Indeed, the following inequality holds for any graph G

$$\theta_E(E_\triangle(G); G) + \max\{\min\{1, |\overline{E}_\triangle(G)|\}, |\overline{E}_\triangle(G)| - |V(G)| + 2\} \leq \theta_E(G).$$

Proof. If $|\overline{E}_\triangle(G)| = 0$, then the left hand side of the above inequality is equal to $\theta_E(E_\triangle(G); G)$ which is less than or equal to $\theta_E(G)$. Now suppose that $|\overline{E}_\triangle(G)| \geq 1$. Then $\min\{1, |\overline{E}_\triangle(G)|\} = 1$ and the left hand side is equal to $\theta_E(E_\triangle(G); G) + 1$ or $\theta_E(E_\triangle(G); G) + |\overline{E}_\triangle(G)| - |V(G)| + 2$. In addition, $|V(G)| \geq 2$. Since $|\overline{E}_\triangle(G)| \geq 1$ and $|V(G)| \geq 2$, both $\theta_E(E_\triangle(G); G) + 1$ and $\theta_E(E_\triangle(G); G) + |\overline{E}_\triangle(G)| - |V(G)| + 2$ are less than or equal to $\theta_E(E_\triangle(G); G) + |\overline{E}_\triangle(G)|$. Thus, the inequality holds by Lemma 2.9. $\square$

As a corollary of Theorem 2.10, we obtain the following result which gives a sufficient condition for graphs to be competitively tight.

Corollary 2.12. *If a graph G satisfies $|\overline{E}_\triangle(G)| \geq |V(G)| - i(G) - 1$ and $i(G) \leq k(G)$ where $i(G)$ is the number of isolated vertices of G, then G is competitively tight.*

Proof. Let G' be the graph obtained by deleting the isolated vertices from G. Since $i(G) \leq k(G)$, it follows from Lemma 2.1 that $k(G) = k(G') - i(G)$. Since $k(G) \geq 0$, we have $i(G) \leq k(G')$. Thus, by Proposition 2.2, it is sufficient to show that G' is competitively tight. Since $|\overline{E}_\triangle(G')| = |\overline{E}_\triangle(G)|$ and $|V(G')| = |V(G)| - i(G)$, we have $|\overline{E}_\triangle(G')| \geq |V(G')| - 1$ and so $|\overline{E}_\triangle(G')| - |V(G')| + 2 \geq 1 \geq \min\{1, |\overline{E}_\triangle(G')|\}$. By Lemma 2.9 and Theorem 2.10, $k(G') \leq \theta_E(E_\triangle(G'); G') + |\overline{E}_\triangle(G')| - |V(G')| + 2 = \theta_E(G') - |V(G')| + 2$. By Theorem 1.1, we obtain $k(G') = \theta_E(G') - |V(G')| + 2$. $\square$

We present a family of graphs satisfying the sufficient condition for a graph being competitively tight. Let t and n be positive integers with $t \geq 3$. Let $G_{t,n}$ be the connected graph defined by

$$\begin{aligned} V(G_{t,n}) &= \{v_1, \ldots, v_{3tn}\}, \\ E(G_{t,n}) &= \{v_i v_{i+1} : 1 \leq i \leq 3tn - 1\} \cup \bigcup_{m=0}^{n-1} \{v_{3tm+3i} v_{3tm+3j} : 1 \leq i < j \leq t\} \end{aligned}$$

(see Figure 2). It is easy to check that $\overline{E}_\triangle(G_{t,n})$ is the Hamilton path $v_1v_2\dots v_{3tn}$ of $G_{t,n}$ and so $|\overline{E}_\triangle(G_{t,n})| = |V(G)| - 1$.

On the other hand, each of the edges on the Hamilton path $v_1v_2\cdots v_{3tn}$ forms a maximal clique. Other than those cliques, $\{v_{3tm+3i}v_{3tm+3j} : 1 \le i < j \le t\}$ is a maximal clique for each m, $0 \le m \le n-1$. It can easily be seen that these maximal cliques form an edge clique cover whose size is minimum among all edge clique covers of $G_{t,n}$, which implies that $\theta_E(G_{t,n}) = (3tn-1)+n$. Thus, by Corollary 2.12,

$$k(G) = (3tn+n-1) - 3tn + 2 = n+1.$$

For any positive integer n, let $G = G_{3,n}$. Then $v_{9i+3}v_{9i+6}v_{9i+9}$ are the only triangles of G $(0 \le i \le n-1)$ and so G has exactly n triangles. As we have shown, it holds that $k(G) = n+1 = \theta_E(G) - |V(G)| + 2$. Hence G is a competitively tight graph with exactly n triangles.

It is also possible that a competitively tight graph has a clique of any size: For any positive integer t with $t \ge 3$, let $G = G_{t,1}$. Then $S = \{v_{3i} \in V : 1 \le i \le t\}$ is a clique of size t of G. As we have shown, it holds that $k(G) = 2 = \theta_E(G) - |V(G)| + 2$. Hence G is competitively tight.

The following gives a necessary condition for graphs to be competitively tight.

Proposition 2.13. *If a graph G is competitively tight, then $|\overline{E}_\triangle(G)| \ge |V(G)| - \theta_E(E_\triangle(G); G) - 2$.*

Proof. Since G is competitively tight, $k(G) = \theta_E(G) - |V(G)| + 2$ holds. By Lemma 2.9, we have $\theta_E(E_\triangle(G); G) + |\overline{E}_\triangle(G)| - |V(G)| + 2 = \theta_E(G) - |V(G)| + 2 = k(G) \ge 0$. Hence $|\overline{E}_\triangle(G)| \ge |V(G)| - \theta_E(E_\triangle(G); G) - 2$. □

It follows from Corollary 2.12 that any graph G having exactly three triangles and having no isolated vertices is competitively tight if it satisfies $|E(G)| \ge |V(G)| + 8$. To see why, note that $|E_\triangle(G)| \le 9$. Since $|\overline{E}_\triangle(G)| = |E(G)| - |E_\triangle(G)|$,

$$|\overline{E}_\triangle(G)| = |E(G)| - |E_\triangle(G)| \ge (|V(G)| + 8) - 9 \ge |V(G)| - i(G) - 1$$

and so, by Corollary 2.12, G is competitively tight.

On the other hand, we know from Proposition 2.13 that a graph G having exactly three triangles is not competitively tight if it satisfies $|E(G)| \le |V(G)| + 1$. To show it, we first note that $\theta_E(E_\triangle(G); G) = 3$ and $7 \le |E_\triangle(G)|$. Then

$$|\overline{E}_\triangle(G)| = |E(G)| - |E_\triangle(G)| \le (|V(G)| + 1) - 7 \le |V(G)| - \theta_E(E_\triangle(G); G) - 3$$

and so, by Proposition 2.13, G is not competitively tight.

3. Further Study

The lower bound given in Corollary 2.12 can be improved. To take a competitively tight graph which does not satisfy the condition of Corollary 2.12, let n and p be integers with $n \ge 7$ and $2 \le p < \lfloor \frac{n}{3} \rfloor$. Let G be the Cayley graph associated with $(\mathbb{Z}/n\mathbb{Z}, \{\pm 1, \pm 2, \dots, \pm p\})$, i.e., G is the graph defined by

$$V(G) = \{v_i : i \in \mathbb{Z}/n\mathbb{Z}\} \quad \text{and} \quad E(G) = \{v_iv_j : i - j \in \{\pm 1, \pm 2, \dots, \pm p\}\}.$$

Then $|V(G)| = n$ and $|\overline{E}_{\triangle}(G)| = 0$. Therefore, $|\overline{E}_{\triangle}(G)| < |V(G)| - 1$. As $i(G) = 0$, G does not satisfy the condition of Corollary 2.12.

Since any two of the edges in $\{v_i v_j : i - j \in \{\pm p\}\}$ are not covered by the same clique in G, any edge clique cover of G contains at least n cliques. Therefore, $\theta_E(G) \geq n$. Since $\theta_E(G) \leq n$ by [5, Lemma 2.4], we have $\theta_E(G) = n$. Note that $k(G) = 2$ by [5, Theorem 1.3]. Thus G is competitively tight.

Accordingly, we propose improving the lower bound given in Corollary 2.12 as a further study. In a similar vein, we suggest finding out whether or not the lower bound given in Proposition 2.13 is sharp.

Acknowledgments

The authors are grateful to the anonymous referees for suggestions leading to improvements in the presentation of the results.

References

[1] J. E. Cohen: Interval graphs and food webs: a finding and a problem, RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.

[2] S. -R. Kim: The competition number and its variants, in *Quo Vadis, Graph Theory?* (J. Gimbel, J. W. Kennedy, and L. V. Quintas, eds.), Annals of Discrete Mathematics **55**, North Holland B. V., Amsterdam, the Netherlands, (1993) 313-326.

[3] S. -R. Kim: The competition number of triangle-free graphs, *Congressus Numerantium* **110** (1995) 97–105.

[4] S. -R. Kim, J. Y. Lee, B. Park, and Y. Sano: The competition number of a graph and the dimension of its hole space, *Applied Mathematics Letters* **25** (2012) 638–642.

[5] S. -R. Kim, B. Park, and Y. Sano: The competition number of the complement of a cycle, *Discrete Applied Mathematics*, Available online (2011). `doi:10.1016/j.dam.2011.10.034`

[6] S. -R. Kim and F. S. Roberts: Competition numbers of graphs with a small number of triangles, *Discrete Applied Mathematics* **78** (1997) 153–162.

[7] B. D. McKay, P. Schweitzer, and P. Schweitzer: Competition numbers, quasi-line graphs and holes, *Preprint.* `arXiv:1110.2933v1`

[8] R. J. Opsut: On the computation of the competition number of a graph, *SIAM Journal on Algebraic and Discrete Methods* **3** (1982) 420–428.

[9] F. S. Roberts: Food webs, competition graphs, and the boxicity of ecological phase space, *Theory and Applications of Graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, Lecture Notes in Mathematics **642**, Springer (1978) 477–490.

[10] Y. Sano: A generalization of Opsut's lower bounds for the competition number of a graph, *Graphs and Combinatorics*, Published online (2012). doi:10.1007/s00373-012-1188-5